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517 (Geometry). Proposed by R. P. BAKER, University of Iowa.

The coördinates of the vertices of a regular icosahedron can be expressed rationally in terms of $(\sqrt{5}-1)/4$ and $\sqrt{(5+\sqrt{5})/8}$, that is, $\cos(2\pi/5)$ and $\sin(2\pi/5)$. Prove (1) that the cosine alone is sufficient; (2) that the irrationalities cannot be reduced further. (The theorem that they cannot be rational is proved in books on crystal theory.)

SOLUTION BY C. F. GUMMER, Queen's University, Kingston.

Since the coördinates of the center are expressible rationally in terms of the vertices, we may suppose it taken as origin. Moreover the vertices are all rational in terms of any three adjacent ones A, B, C; for a fourth is the reflection of A for the plane OBC, and so on. Suppose first that A, B, C have coördinates (o, o, r), $(r\sin\alpha, o, r\cos\alpha)$, $(r\sin\alpha\cos(2\pi/5), r\sin\alpha\sin(2\pi/5), r\cos\alpha)$. Since $AB = BC$, we find $\cos\alpha = 1/\sqrt{5}$; and α is the angle subtended at O by an edge. Hence, the general equations for A, B, C (with the origin for center) are

$$x_1^2+y_1^2+z_1^2 = x_2^2+y_2^2+z_2^2 = x_3^2+y_3^2+z_3^2 = \sqrt{5}(x_1x_2+y_1y_2+z_1z_2)$$
$$= \sqrt{5}(x_2x_3+y_2y_3+z_2z_3) = \sqrt{5}(x_3x_1+y_3y_1+z_3z_1).$$

Evidently no rational solution exists, which proves the second part of the theorem.

A simple solution is found by assuming $x_1 = y_2 = z_3 = u$, $x_2 = y_3 = z_1 = v$, $x_3 = y_1 = z_2 = w$, so that the equations become $u^2+v^2+w^2 = \sqrt{5}(uv+vw+wu)$. If $w = 0$, $u^2+v^2 = \sqrt{5}\,uv$, which is satisfied by $u = 1$, $v = 2\cos(2\pi/5)$. Therefore, A, B, C may be taken to be $(1, 0, 2\cos(2\pi/5))$, $(2\cos(2\pi/5), 1, 0)$, $(0, 2\cos(2\pi/5), 1)$.

518 (Geometry). Proposed by ROGER A. JOHNSON, Cleveland, Ohio.

If one angle of a triangle is 60°, the Euler line (the line through the circumcenter, orthocenter, and median point) is perpendicular to the bisector of that angle; and if one angle is 120°, the Euler line is parallel to the bisector of that angle.

SOLUTION BY J. L. COOLIDGE, Harvard University.

Let the vertices of the triangle be A_1, A_2, A_3, the middle points of the sides M_1, M_2, M_3, the feet of the altitudes H_1, H_2, H_3, the circumcenter O, and the orthocenter H.

Let us take A_1 as the angle in which we are interested, and assume it first to be acute.

The angle which the external bisector forms with A_2A_3 is $\frac{1}{2}(A_2 - A_3)$ and its tangent

$$\tan\tfrac{1}{2}(A_2 - A_3) = \frac{1-\cos(A_2 - A_3)}{\sin(A_2 - A_3)}.$$

The tangent of the angle which the Euler line makes with A_2A_3 is

$$\frac{OM_1 - HH_1}{A_2M_1 - A_2H_1}.$$

We have

$$OM_1 = r\cos A_1 = r[\sin A_2\sin A_3 - \cos A_2\cos A_3],$$
$$A_2M_1 = r\sin A_1 = r[\sin A_2\cos A_3 + \cos A_2\sin A_3],$$
$$HH_1 = (A_2H_1)\operatorname{ctn} A_3 = 2r\cos A_2\cos A_3,$$

and

$$A_2H_1 = a_3\cos A_2 = 2r\cos A_2\sin A_3.$$

The tangent of this angle is

$$\frac{\sin A_2\sin A_3 - 3\cos A_2\cos A_3}{\sin(A_2 - A_3)}.$$

The two tangents are equal if

$$1 - \cos A_2\cos A_3 - \sin A_2\sin A_3 = \sin A_2\sin A_3 - 3\cos A_1\cos A_3, \text{ or } \cos(A_2 + A_3) = -\tfrac{1}{2}.$$

Hence,

$$A_2 + A_3 = 120^\circ.$$

The case when A_1 is obtuse may be treated by a similar method.

Also solved by A. M. HARDING, NATHAN ALTSHILLER, C. C. YEN, J. F. LÜ, K. K. CHAN, HORACE OLSON, G. BREIT, H. C. GOSSARD, FRANK V. MORLEY, LOUIS WEISNER, and OTTO J. RAMLER.

428 (Calculus). Proposed by J. L. RILEY, Northwestern State Normal School, Tahlequah, Okla.

The loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{a}{y}\right)^{4/3},$$

and that $2\rho\rho' = a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

SOLUTION BY A. M. HARDING, University of Arkansas.

The problem, as originally stated, is incorrect. It should read as above. The equation of the lemniscate, referred to a tangent at its center, is $r^2 = a^2 \sin 2\theta$. By the formulas of the elementary calculus it is easily shown that

$$s = \int_0^\theta a\sqrt{\csc 2\theta}d\theta, \qquad \psi = 2\theta, \qquad \text{and} \qquad \rho = \frac{a}{3}\sqrt{\csc 2\theta},$$

where s is the length of arc from the node $A(x, y)$ to the point P, ψ is the angle between AP and the tangent at P, and ρ is the radius of curvature at P.

Let the lemniscate roll along the x-axis until the tangent at P coincides with this axis, and let O be the origin of coördinates. Then

$$x = OP - AP\cos\psi = a\int_0^\theta \sqrt{\csc 2\theta}d\theta - a\sqrt{\sin 2\theta}\cos 2\theta, \text{ and } y = AP\sin\psi = a\sqrt{\sin 2\theta}\sin 2\theta;$$

whence,

$$\frac{dy}{dx} = \cot 2\theta,$$

and

$$\frac{d^2y}{dx^2} = -\frac{2(\csc 2\theta)^{3/2}}{3a\sin^2 2\theta}.$$

Hence,

$$1 + \left(\frac{dy}{dx}\right)^2 = \csc^2 2\theta = \left(\frac{a}{y}\right)^{4/3}$$

and

$$\rho' = -\frac{3}{2}a\sqrt{\sin 2\theta}.$$

Hence,

$$\rho\rho' = -\frac{a^2}{2}.$$

Also solved by WILLIAM HOOVER.

430 (Calculus). Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z-axis and pass a plane through the x- (or y-) axis. Find the area of this surface intercepted by this plane and the xz- (or yz-) plane.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let a = the radius of the circle, c = the distance of the chord from the center, 2α = the angle subtended at the center by the arc of the chord; take the middle of the chord for the origin, and the radius at right angles to the chord for the x-axis; the equation to the generating arc is then